Problem: Sorting

- arranging elements of set into order
- Algorithm design technique:
  - **Divide and Conquer**
- Solution:
  - **Insertion Sort**
  - **Quicksort**
  - **Mergesort**
  - **Heapsort**
  - **Shellsort**
  - **Radix Sorting**
- Optimality:
  - **Lower bounds for Sorting by Comparison of Keys**

TECH
Computer Science

Application of Sorting

- For searching on unsorted data by comparing keys, optimal solutions require $\theta(n)$ comparisons.
- For searching on sorted data by comparing keys, optimal solutions require $\theta(\log n)$ comparisons.
- Sorting data for users
- More…

Insertion Sort

- Strategy:
  - **Insertion of an element in proper order:**
  - **Begin with a sequence E of n elements in arbitrary order**
  - **Initially assume the sorted segment contains first element**
  - **Let x be the next element to be inserted in sorted segment, pull x "out of the way", leaving a vacancy**
  - **repeatedly compare x to the element just to the left of the vacancy, and as long as x is smaller, move that element into the vacancy,**
  - **else put x in the vacancy,**
  - **repeat the next element that has not yet examined.**

Insertion Sort: Algorithm

- Input
  - **E, an array of elements, and n >=0, the number of elements. The range of indexes is 0, …, n-1**
- Output
  - **E, with elements in nondecreasing order of their keys**
- void insertionSort(Element[] E, int n)
  - **int xindex;**
  - **for (xindex = 1; xindex < n; xindex++)**
    - Element current = E[xindex];
    - key x = current.key
    - int xLoc = shiftVacRec(E, xindex, x);
    - E[xLoc] = current;
  - **return;**

Insertion Sort: Specification for subroutine

- Specification
  - **int shiftVacRec(Element[] E, int vacant, Key x)**
  - **Precondition**
    - Vacant is nonnegative
  - **Postconditions**
    - 1. Elements in E at indexes less than xLoc are in their original positions and have keys less than or equal to x
    - 2. Elements in E at positions xLoc+1, …, vacant are greater than x and were shifted up by one position from their positions when shiftVacRec was invoked.

Insertion Sort: Algorithm shiftVacRec

- int shiftVacRec(Element[] E, int vacant, Key x)
  - **int xLoc;**
  - **if (vacant == 0)**
    - xLoc = vacant;
  - **else if (E[vacant-1].key <= x)**
    - xLoc = vacant;
  - **else**
    - E[vacant] = E[vacant-1];
    - xLoc = shiftVacRec(E, vacant-1, x);
  - **return xLoc**

Insertion Sort: Analysis

- Worst-Case Complexity
  - $W(n) = \sum\{i=1 \text{ to } n-1\}[i] = n(n-1)/2 \in \theta(n^2)$
- Average Behavior
  - **average number of comparisons in shiftVacRec**
  - $1/(i+1) \sum\{i=1 \text{ to } j\} (j) + i/(i+1) = i/2+1-1/(i+1)$
  - $A(n) = \sum\{i=1 \text{ to } n-1\} [1/2+1-1/(i+1)] \approx (n^2)/4$

Insertion Sort: Optimality

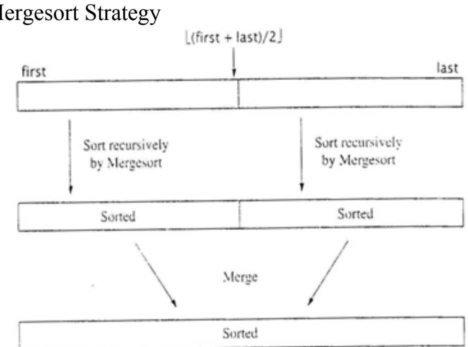
- Theorem 4.1
  - **Any algorithm that sorts by comparison of keys and removes at most one inversion after each comparison must do at least n(n-1)/2 comparisons in the worst case and at least n(n-1)/4 comparisons on the average (for n elements)**
- Proof…
- Insertion Sort is optimal for algorithms that works "locally" by interchanging only adjacent elements.
- But, it is not the best sorting algorithm.

Algorithm Design Technique: Divide and Conquer

- **It is often easier to *solve several small instances* of a problem than one large one.**
  - *divide* the problem into smaller instances of the same problem
  - solve (*conquer*) the smaller instances recursively
  - *combine* the solutions to obtain the solution for original input
- Solve(I)
  - **n = size(I)**
  - **if (n <= smallsize)**
    - solution = directlySolve(I);
  - **else**
    - divide I into I1, …, Ik.
    - for each i in {1, …, k}
      - Si = solve(Ii);
    - solution = combine(S1, …, Sk);
  - **return solution;**

Using Divide and Conquer: Mergesort

- Mergesort Strategy



Algorithm: Mergesort

- **Input: Array E and indexs first, and Last, such that the elements E[i] are defined for first <= i <= last.**
- **Output: E[first], …, E[last] is sorted rearrangement of the same elements**
- void mergeSort(Element[] E, int first, int last)
  - **if (first < last)**
    - int mid = (first+last)/2;
    - mergeSort(E, first, mid);
    - mergeSort(E, mid+1, last);
    - merge(E, first, mid, last);
  - **return;**
- $W(n) = W(n/2)+W(n/2) + Wmerge(n) \in \theta(n \log n)$
  - **Wmerge(n) = n-1**
  - **W(1) = 0**

Merging Sorted Sequences

- Problem:
  - **Given two sequences A and B sorted in nondecreasing order, merge them to create one sorted sequence C**
- Strategy:
  - **determine the first item in C: It is the minimum between the first items of A and B. Suppose it is the first items of A. Then, rest of C consisting of merging rest of A with B.**

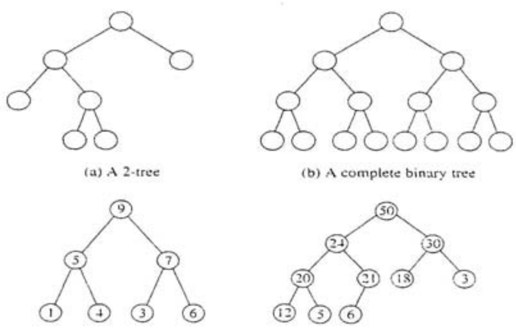
Algorithm: Merge

- Merge(A, B, C)
  - **if (A is empty)**
    - rest of C = rest of B
  - **else if (B is empty)**
    - rest of C = rest of A
  - **else if (first of A <= first of B)**
    - first of C = first of A
    - merge (rest of A, B, rest of C)
  - **else**
    - first of C = first of B
    - merge (A, rest of B, rest of C)
  - **return**
- W(n) = n – 1

Heap and Heapsort

- A Heap data structure is a binary tree with special properties:
  - **Heap Structure**
  - **Partial order tree property**
- Definition: Heap Structure
  - **A binary tree T is a heap structure if and only if it satisfies the following conditions: (h = height of the tree)**
    - 1. T is complete at least through depth h-1
    - 2. All leaves are at depth h or h –1
    - 3. All paths to leaf of depth h are to the left of all parts to a leaf of depth h-1
  - **Such a tree is also called a left-complete binary tree.**
- Definition: Partial order tree property
  - **A tree T is a (maximizing) partial order tree if and only if the key at any node is greater than or equal to the keys at each of its children (if it has any)**

e.g. Heaps (or not)

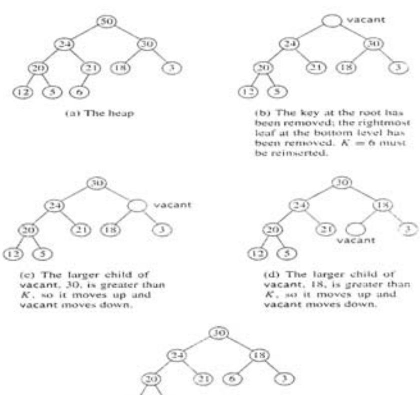


(a) A 2-tree (b) A complete binary tree (c) Heap 1 (d) Heap 2

Heapsort Strategy

- If the elements to be sorted are arranged in a heap,
- then we can build a sorted sequence in reverse order by repeatedly removing the element from the root,
- rearranging the remaining elements to reestablish the partial order tree property, and so on.
- How does it work?

Heapsort in action



(a) The heap

(b) The key at the root has been removed; the rightmost leaf at the bottom level has been removed. $K = 6$ must be reinserted.

(c) The larger child of vacant, 30, is greater than K, so it moves up and vacant moves down.

(d) The larger child of vacant, 18, is greater than K, so it moves up and vacant moves down.

(e) Finally, since vacant is a leaf, $K = 6$ is inserted.

Heapsort Outlines

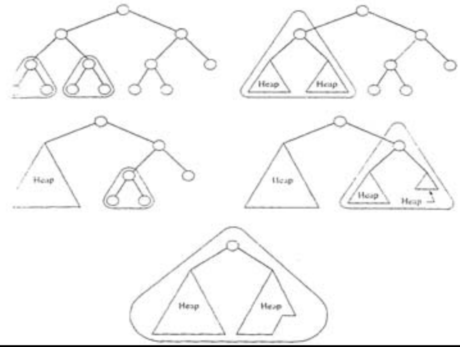
- heapSort(E, n) // Outline
  - **construct H from E, the set of n elements to be sorted**
  - **for (i = n; i >= 1; i--)**
    - curMax = getMax(H)
    - deleteMax(H);
    - E[i] = curMax;
- deteleMax(H) // Outline
  - **copy the rightmost element of the lowest level of H into K**
  - **delete the rightmost element on the lowest level of H**
  - **fixHeap(H, K);**

Fixheap Outline

- fixHeap(H, K) // Outline
  - **if (H is a leaf)**
    - insert K in root(H);
  - **else**
    - set largerSubHeap to leftSubtree(H) or rightSubtree(H), whichever has larger key at is root. This involves one key comparison.
    - if (K.key >= root(largerSubHeap.key)
      - insert K in root(H);
    - else
      - insert root(largerSubHeap) in root(H);
      - fixHeap(largerSubHeap, K);
  - **return;**
- FixHeap requires 2h comparisons of keys in the worst case on a heap with height h. $W(n) \approx 2 \lg(n)$

Heap construction Strategy (divide and conquer)

- base case is a tree consisting of one node



Construct Heap Outline

- **Input: A heap structure H that does not necessarily have the partial order tree property**
- **Output: H with the same nodes rearranged to satisfy the partial order tree property**
- void constructHeap(H) // Outline
  - **if (H is not a leaf)**
    - constructHeap (left subtree of H);
    - constructHeap (right subtree of H);
    - Element K = root(H);
    - fixHeap(H, K);
  - **return;**
- $W(n) = W(n-r-1) + W(r) + 2 \lg(n)$ for $n > 1$
- $W(n) \in \theta(n)$; heap is constructed in linear time.

Heapsort Analysis

- The number of comparisons done by fixHeap on heap with k nodes is at most 2 lg(k)
  - **so the total for all deletions is at most**
  - **2Σ [k=1 to n-1](lg(k)) $\in \theta(2n \lg(n))$**
- Theorem: The number of comparisons of keys done by Heapsort in the worst case is $2n \lg(n) + O(n)$.
- Heapsort does $\theta(n \lg(n))$ comparisons on average as well. (How do we know this?)

Implementation issue: storing a tree in an array

- Array E with range from 1, …, n
- Suppose the index i of a node is given, then
  - **left child has index 2i**
  - **right child has index 2i + 1**
  - **parent has index floor(i/2)**
- e.g.

9	5	7	1	4	3	6

Heap 1

50	24	30	20	21	18	3	12	5	6

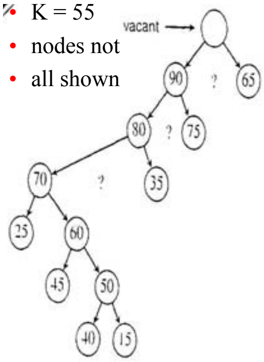
Heap 2

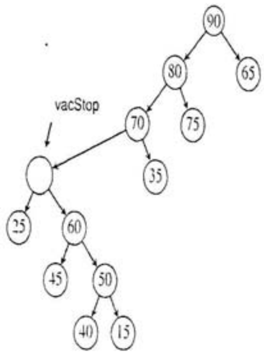
Accelerated Heapsort

- Speed up Heapsort by about a factor of two.
- Normal fixHeap costs 2h comparisons in the worst case. Can we do better?
- …
- The solution is a surprising application of divide and conquer!
  - **filter the vacant position halfway down the tree, h/2**
  - **test whether K is bigger than the parent of vacant**
  - **yes: bubble the vacant back up to where K should be**
  - **no: repeat filter the vacant position another halfway down *recursively*!**

Accelerated Heapsort Strategy in Action

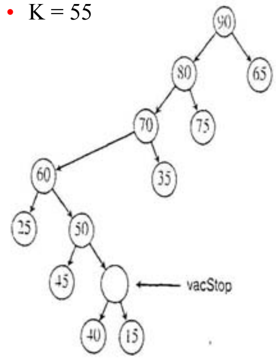
- K = 55
- nodes not
- all shown

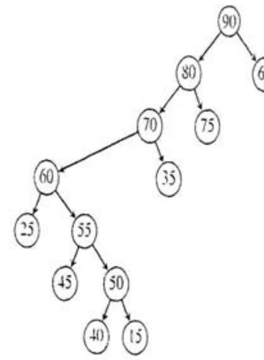




Action continues

- K = 55





Accelerated Heapsort Algorithm

- void fixHeapFast(Element[] E, int n, Element K, int vacant, int h)
  - **if (h <= 1)**
    - Process heap of height 0 or 1
  - **else**
    - int hStop = h/2
    - int vacStop = **promoste** (E, hStop, vacant, h);
    - // vacStop is new vacnt location, at height hStop
    - int vacParent = vacStop / 2;
    - if (E[vacParent].key <= K.key)
      - E[vacStop] = E[vacParent];
      - **bubbleUpHeap** (E, vacant, K, vacParent);
    - else
      - fixHeapFast (E, n, K, vacStop, hStop);

Algorithm: promote

- int promote (Element[] E, int hStop, int vacant, int h)
  - **int vacStop;**
  - **if (h <= hStop)**
    - vacStop = vacant;
  - **else if (E[2*vacant].key <= E[2*vacant+1].key)**
    - E[vacant] = E[2*vacant+1];
    - vacStop = promote (E, hStop, 2*vacant+1, h-1)
  - **else**
    - E[vacant] = E[2*vacant];
    - vacStop = promote (E, hStop, 2*vacant, h-1);
  - **return vacStop;**

Algorithm: bubbleUpHeap

- void bubbleUpHeap (Element[] E, int root, Element K, int vacant)
  - **if (vacant == root)**
    - E[vacant] = K;
  - **else**
    - int parent = vacant / 2;
    - if (K.key <= E[parent].key)
      - E[vacant] = K;
    - else
      - E[vacant] = E[parent];
      - bubbleUpHeap (E, root, K, parent);

Analysis: fixHeapFast

- Essentially, there is one comparison each time vacant changes a level due to the action of either bubbleUpHeap or Promote. The total is h.
- Assume bubbleUpHeap is never call, so fixHeapFast reaches its base case. Then, it requires lg(h) checks along the way to see whether it needs to reverse direction.
- Therefore, altogether fixHeapFast uses h+lg(h) comparisons in the worst case.

Accelerated Heapsort Analysis

- The number of comparisons done by fixHeapFast on heap with k nodes is at most lg(k)
  - **so the total for all deletions is at most**
  - **Σ [k=1 to n-1](lg(k)) $\in \theta$(n lg(n))**
- Theorem: The number of comparisons of keys done by Accelerated Heapsort in the worst case is n lg(n) + O(n).

Comparison of Four Sorting Algorithms

Algorithm	Worst case	Average	Space Usage
Insertion	n^2/2	θ(n^2)	in place
Quicksort	n^2/2	θ(n log n)	log n
Mergesort	n lg n	θ(n log n)	n
Heapsort	2n lg n	θ(n log n)	in place
Ac.Heaps.	n lg n	θ(n log n)	in place

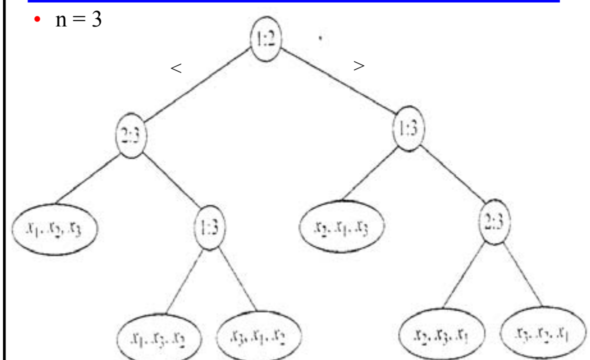
- Accelerated Heapsort currently is the method of choice.

Lower Bounds for Sorting by Comparison of Keys

- The Best possible!
  - **Lower Bound for Worst Case**
  - **Lower Bound for Average Behavior**
- Use decision tree for analyzing the class of sorting algorithms (by comparison of keys)
  - **Assuming the keys in the array to be sorted are distinct.**
  - **Each internal node associates with one comparison for keys x_i and x_j; labeled i :j**
  - **Each leaf nodes associates with one permutation (total n! permutations for problem size n)**
  - **The action of Sort on a particular input corresponds to following one path in its decision tree from the root to a leaf.**

Decision tree for sorting algorithms

- n = 3



Lower Bound for Worst Case

- Lemma:
  - **Let L be the number of leaves in a binary tree and let h be its height.**
  - **Then L <= 2^h, and h >= Ceiling[lg L]**
  - **For a given n, L = n!, the decision tree for any algorithm that sorts by comparison of keys has height as least Ceiling [lg n!].**
- Theorem:
  - **Any algorithm to sort n items by comparisons of keys must do at least Ceiling [lg n!],**
  - **or approximately Ceiling [n lg n – 1.443 n],**
  - **key comparisons in the worst case.**

Lower Bound for Average Behavior

- Theorem:
  - **The average number of comparisons done by an algorithm to sort n items by comparison of keys is at least lg n!**
  - **or approximately n lg n – 1.443 n**
- The only difference from the worst-case lower bound is that there is no rounding up to an integer
  - **the average needs not be an integer,**
  - **but the worst case must be.**

Improvement beyond lower bound?! Know more → Do better

- Up to now,
  - only one assumption was make about the keys: They are elements of linearly ordered set.
  - The basic operation of the algorithms is a comparison of two keys.
- If we *know more* (or make more assumptions) about the keys,
  - we can consider algorithms that perform other operations on them.
    - // Recall algorithms for searching from unordered data vs. searching from ordered data

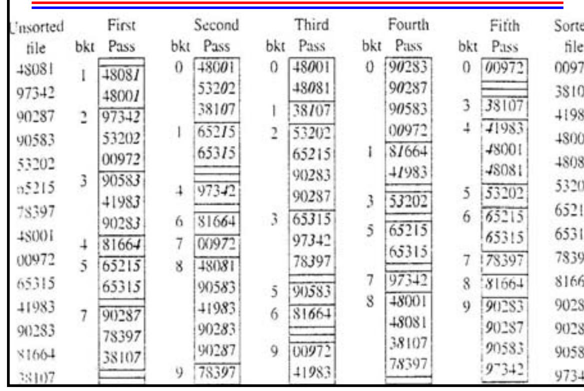
Using properties of the keys

- support the keys are names
- support the keys are all five-digit decimal integers
- support the keys are integer between 1 and m.
- For sorting each of these examples, the keys are
  - distributed into different piles as a result of examining individual letters or digits in a key or comparing keys to predetermined values
  - sort each pile individually
  - combine all sorted piles
- Algorithms that sort by such methods are not in the class of algorithms previously considered because
  - to use them we must know something about either the structure or the range of the keys.

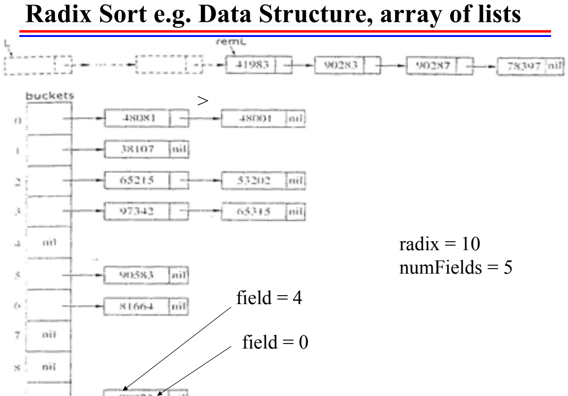
Radix Sort

- Strategy: It is *startling* that
  - if the keys are distributed into piles (also called buckets) first according to their ***least significant*** digits (or bits, letters, or fields),
    - and the piles are combined in order
    - and the relative order of two keys placed in the same pile is not changed
  - then the problem of sorting the piles has been completely eliminated!

Radix Sort e.g Start from least significant digit



Radix Sort e.g. Data Structure, array of lists



Radix Sort: Algorithm

- List radixSort (List L, int radix, int numFields)
  - List[] buckets = new List[radix];
  - int field; // filed number within the key
  - List newL;
  - newL = L;
  - For (filed = 0; field < numFields; field++)
    - Initialize buckets array to empty lists.
    - distribute (newL, buckets, radix, field);
    - newL = combine (buckets, radix);
  - return newL;

Radix Sort: distribute

- void distribute (List L, List[] buckets, int radix, int field)
  - **//distribute keys into buckets**
  - **List remL;**
  - **remL = L;**
  - **while (remL != nil)**
    - Element K = first (remL);
    - int b = maskShift (field, radix, K.key);
      - // maskShif(f, r, key) selects field f (counting from the right) of key,
      - // based on radix r. the result, b, is the range 0 … radix –1,
      - // and is the bucket number for K
    - buckets[b] = cons(K, buckets[b]); // construct list
    - remL = rest (remL);
  - **return**

Radix Sort: Combine

- List combine (List[] buckets, int radix)
  - **// Combine linked lists in all buckets into one list L**
  - **int b; // bucket number**
  - **List L, remBucket;**
  - **L = nil;**
  - **for (b = radix-1; b>=0; b--)**
    - remBucket = buckets[b];
    - while (remBucket != nil)
      - key K = first (remBucket);
      - L = cons (K, L);
      - remBucket = rest (remBucket);
  - **return L;**

Radix Sort: Analysis

  - **distribute does θ(n) steps**
  - **combine does θ(n) steps**
- if number of field is constant,
  - **then the total number of steps done by radix sort is *linear in n*.**
- radix sort use θ(n) extra space for link fields, provided the radix is bounded by n.

Exercise

- Do it or loss it!