Graphs and Graph Traversals

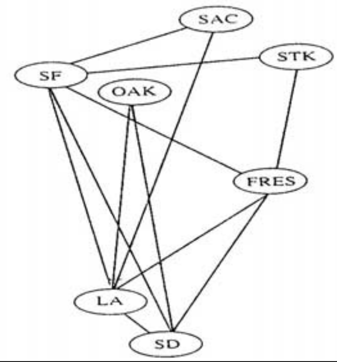
- // From Tree to Graph
- // Many programs can be cast as problems on graph

- Definitions and Representations
- Traversing Graphs
- Depth-First Search on Directed Graphs
- Strongly Connected Components of a Directed Graph
- Depth-First Search on Undirected Graphs
- Biconnected Components of an Undirected Graph

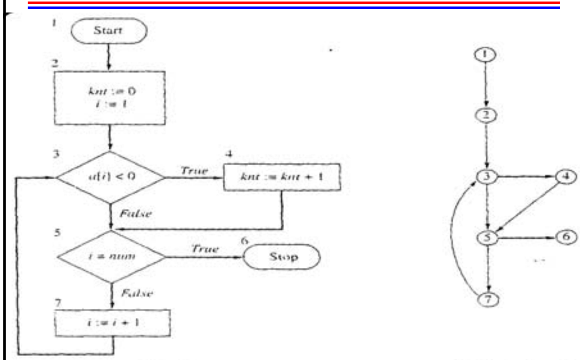
TECH

Computer Science

Problems: e.g. Airline Routes



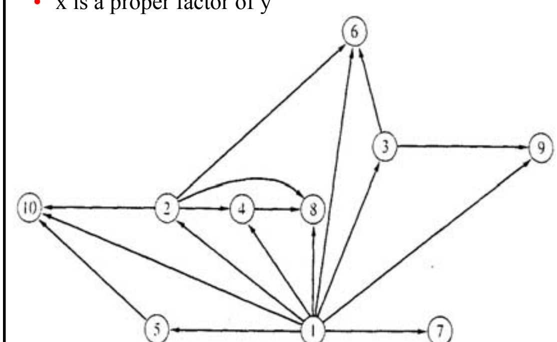
Problems: e.g. Flowcharts



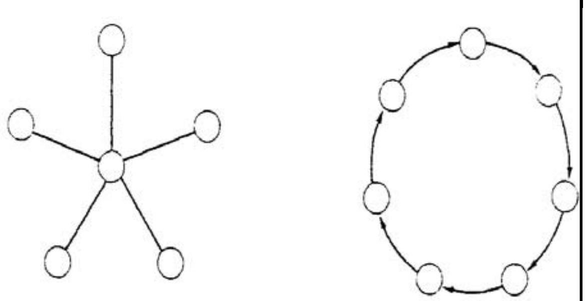
(a) Flowchart (b) Directed graph

Problems: e.g. Binary relation

- x is a proper factor of y



Problems: e.g. Computer Networks



(a) A star network (b) A ring network

Definition: Directed graph

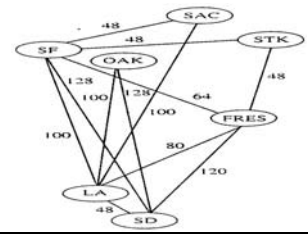
- Directed Graph
  - **A directed graph, or digraph, is a pair**
  - **G = (V, E)**
  - **where V is a set whose elements are called vertices, and**
  - **E is a set of ordered pairs of elements of V.**
    - Vertices are often also called nodes.
    - Elements of E are called edges, or directed edges, or arcs.
    - For directed edge (v, w) in E, v is its tail and w its head;
    - (v, w) is represented in the diagrams as the arrow, v -> w.
    - In text we simple write vw.

Definition: Undirected graph

- Undirected Graph
  - **A undirected graph is a pair**
  - **G = (V, E)**
  - **where V is a set whose elements are called vertices, and**
  - **E is a set of *unordered* pairs of *distinct* elements of V.**
    - Vertices are often also called nodes.
    - Elements of E are called edges, or undirected edges.
    - Each edge may be considered as a subset of V containing two elements,
    - {v, w} denotes an undirected edge
    - In diagrams this edge is the line v---w.
    - In text we simple write vw, or wv
    - vw is said to be *incident* upon the vertices v and w

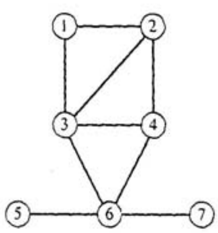
Definitions: Weighted Graph

- A weighted graph is a triple (V, E, W)
  - **where (V, E) is a graph (directed or undirected) and**
  - **W is a function from E into R, the reals (integer or rationals).**
  - **For an edge e, W(e) is called the weight of e.**



Graph Representations using Data Structures

- Adjacency Matrix Representation
  - **Let G = (V,E), n = |V|, m = |E|, V = {v1, v2, …, vn)**
  - **G can be represented by an n × n matrix**

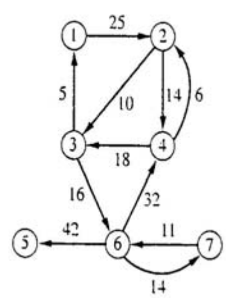


(a) An undirected graph

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(b) Its adjacency matrix

Adjacency Matrix for weight digraph



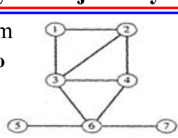
(a) A weighted digraph

$$\begin{pmatrix} 0 & 25.0 & \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & 10.0 & 14.0 & \infty & \infty & \infty \\ 5.0 & \infty & 0 & \infty & \infty & 16.0 & \infty \\ \infty & 6.0 & 18.0 & 0 & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & 32.0 & 42.0 & 0 & 14.0 \\ \infty & \infty & \infty & \infty & \infty & 11.0 & 0 \end{pmatrix}$$

(b) Its adjacency matrix

Array of Adjacency Lists Representation

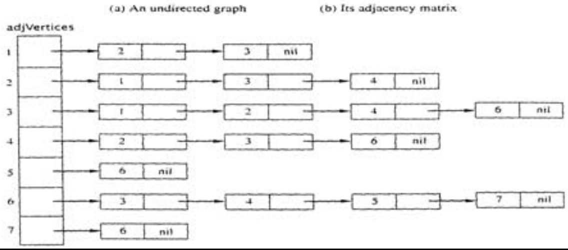
- From
  - **to**



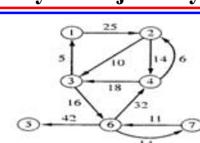
(a) An undirected graph

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(b) Its adjacency matrix



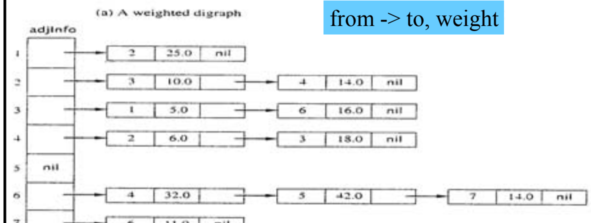
Array of Adjacency Lists Representation



(a) A weighted digraph

$$\begin{pmatrix} 0 & 25.0 & \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & 10.0 & 14.0 & \infty & \infty & \infty \\ 5.0 & \infty & 0 & \infty & \infty & 16.0 & \infty \\ \infty & 6.0 & 18.0 & 0 & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty \\ \infty & \infty & \infty & 32.0 & 42.0 & 0 & 14.0 \\ \infty & \infty & \infty & \infty & \infty & 11.0 & 0 \end{pmatrix}$$

from -> to, weight



More Definitions

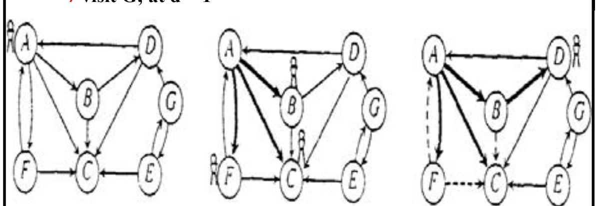
- Subgraph
- Symmetric digraph
- complete graph
- Adjacency relation
- Path, simple path, reachable
- Connected, Strongly Connected
- Cycle, simple cycle
- acyclic
- undirected forest
- free tree, undirected tree
- rooted tree
- Connected component

Traversing Graphs

- Most algorithms for solving problems on a graph examine or process each vertex and each edge.
- Breadth-first search and depth-first search
  - **are two traversal strategies that provide an efficient way to "visit" each vertex and edge edge exactly once.**
- Breadth-first search: Strategy (for digraph)
  - **choose a starting vertex, distance d = 0**
  - **vertices are visited in order of increasing distance from the starting vertex,**
  - **examine all edges leading from vertices (at distance d) to adjacent vertices (at distance d+1)**
  - **then, examine all edges leading from vertices at distance d+1 to distance d+2, and so on,**
  - **until no new vertex is discovered**

Breath-first search, e.g.

- e.g. Start from vertex A, at d = 0
  - **visit B, C, F; at d = 1**
  - **visit D; at d = 2**
- e.g. Start from vertex E, at d = 0
  - **visit G; at d = 1**



Breadth-first search: I/O Data Structures

Input: $G = (V, E)$, a graph represented by an adjacency list structure, adjVertices, as described in Section 7.2.3, where $V = \{1, \ldots, n\}$; $s \in V$, the vertex from which the search begins.

Output: A breadth-first spanning tree, stored in the parent array. The parent array is passed in and the algorithm fills it.

Remarks: For a queue Q, we assume operations of the Queue abstract data type (Section 2.4.2) are used. The array color[1], ..., color[n] denotes the current search status of all vertices. Undiscovered vertices are white; those that are discovered but not yet processed (in the queue) are gray; those that are processed are black.

Breadth-first search: Algorithm

```
void breadthFirstSearch(IntList[] adjVertices, int n, int s, int[] parent)
    int[] color = new int[n+1];
    Queue pending = create(n);
    Initialize color[1], . . ., color[n] to white.

    parent[s] = -1;
    color[s] = gray;
    enqueue(pending, s);
    while (pending is nonempty)
        v = front(pending);
        dequeue(pending);
        For each vertex w in the list adjVertices[v]:
            if (color[w] == white)
                color[w] = gray;
                enqueue(pending, w);
                parent[w] = v;  // Process tree edge vw.
            // Continue through list.
        // Process vertex v here.
        color[v] = black;
    return;
```

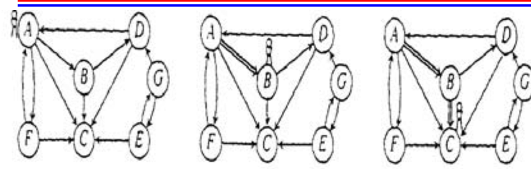
Breadth-first search: Analysis

- For a digraph having n vertices and m edges
  - **Each edge is processed once in the while loop for a cost of θ(m)**
  - **Each vertex is put into the queue once and removed from the queue and processed once, for a cost θ(n)**
  - **Extra space is used for color array and queue, there are θ(n)**
- From a *tree* (breadth-first spanning tree)
  - **the path in the tree from start vertex to any vertex contains the *minimum* possible number of edges**
- Not all vertices are necessarily reachable from a selected starting vertex

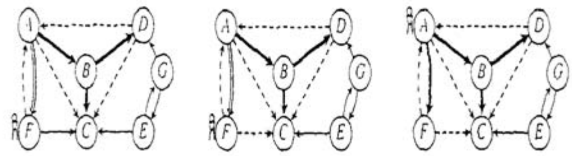
Depth-first search for Digraph

- Depth-first search: Strategy (for digraph)
  - choose a starting vertex, distance d = 0
  - vertices are visited in order of increasing distance from the starting vertex,
  - examine One edges leading from vertices (at distance d) to adjacent vertices (at distance d+1)
  - then, examine One edges leading from vertices at distance d+1 to distance d+2, and so on,
  - until no new vertex is discovered, or dead end
  - then, backtrack one distance back up, and try other edges, and so on
  - until finally backtrack to starting vertex, with no more new vertex to be discovered.

Depth-first search for digraph, e.g. from a tree, remember where we have been

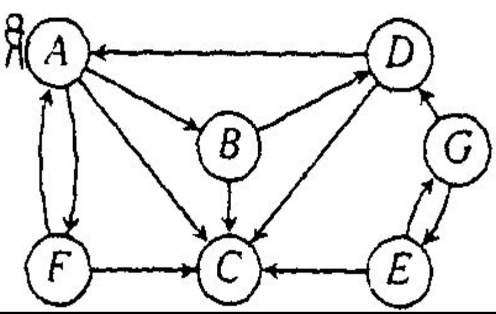


and so on…



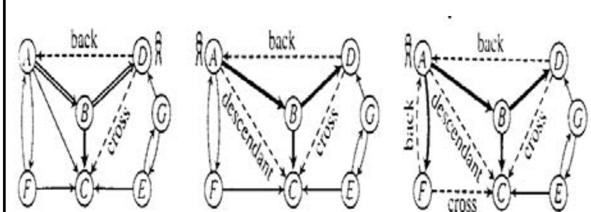
Depth-first Search, e.g. trace it, in order

- Vertex status: undiscovered, discovered, finished
- Edge status: exploring, backtrack, checked



Depth-first search tree

- edges classified:
  - tree edge, back edge, descendant edge, and cross edge



Depth-first search algorithm: outline

dfs(G, v) // *OUTLINE*
    Mark *v* as "discovered."
    For each vertex *w* such that edge *vw* is in *G*:
        If *w* is undiscovered:
            dfs(G, w); that is, explore *vw*, visit *w*, explore from there as much as possible, and backtrack from *w* to *v*.
        Otherwise:
            "Check" *vw* without visiting *w*.
    Mark *v* as "finished."

Reaching all vertices

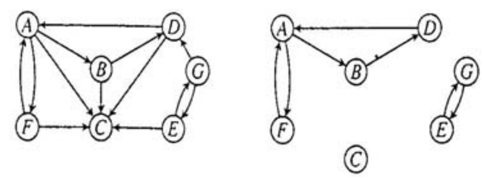
dfsSweep(G) // *OUTLINE*
    Initialize all vertices of *G* to "undiscovered."
    For each vertex $v \in G$, in some order:
        If *v* is undiscovered:
            dfs(G, v); that is, perform a depth-first search beginning (and ending) at *v*; any vertices discovered during an earlier depth-first search visit are not revisited; all vertices visited during this dfs are now classified as "discovered."

Depth-first search algorithm

```
int dfs(IntList[] adjVertices, int[] color, int v, . . .)
    int w;
    IntList remAdj;
    int ans;
    color[v] = gray;
    Preorder processing of vertex v
    remAdj = adjVertices[v];
    while (remAdj ≠ nil)
        w = first(remAdj);
        if (color[w] == white)
            Exploratory processing for tree edge vw
            int wAns = dfs(adjVertices, color, w, . . .);
            Backtrack processing for tree edge vw, using wAns (like inorder)
        else
            Checking (i.e., processing) for nontree edge vw
        remAdj = rest(remAdj)
    Postorder processing of vertex v, including final computation of ans
    color[v] = black;
    return ans;
```

Strongly Connected Components of a Digraph

- Strongly connected:
  - A directed graph is strongly connected if and only if, for each pair of vertices v and w, there is a path from v to w.
- Strongly connected component:
  - A strongly connected component of a digraph G is a maximal strongly connected subgraph of G.



Strongly connected Components and Equivalence Relations

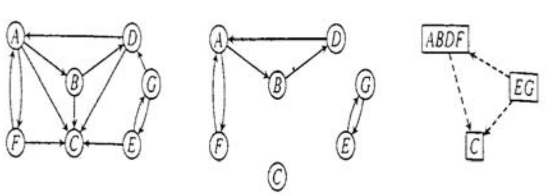
- Strongly Connected Components may be defined in terms of an equivalence relation, S, on the vertices
  - **vSw iff there is a path from v to w and**
  - **a path from w to v**
- Then, a strongly connected component consists of one equivalence class, C, along with all edges vw such that v and w are in C.

Condensation graph

- **The strongly connected components of a digraph can each be collapsed to a single vertex yielding a new digraph that has no cycles.**
- Condensation graph:
  - Let $S_1, S_2, \ldots S_p$ be the strong components of G.
  - The condensation graph of G denoted as G↓, is the digraph G↓ = (V',E'),
  - where V' has p elements $s_1, s_2, \ldots s_p$ and
  - $s_i s_j$ is in E' if and only if i≠j and
  - there is an edge in E from some vertex in S_i to some vertex in S_j.

Condensation graph and its strongly connected components

- Condensation Graph is acyclic.

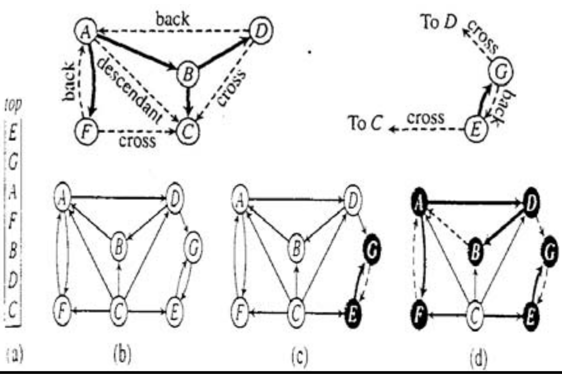


(a) The digraph (b) Its strong components (c) Its condensation graph

Algorithm to Find Strongly Connected Component

- Strategy:
  - **Phase 1:**
  - **A standard depth-first search on G is performed,**
  - **and the vertices are put in a stack at their finishing times**
  - **Phase 2:**
  - **A depth-first search is performed on G^T, the transpose graph.**
  - **To start a search, vertices are popped off the stack.**
  - **A strongly connected component in the graph is identified by the name of its starting vertex (call leader).**

The strategy in Action, e.g.



Depth-First Search on Undirected Graphs

- Depth-first search on an undirected graph is complicated by the fact that edges should by *explored in one direction only*,
- but they are represented twice in the data structure (*symmetric digraph equivalence*)
- Depth-first search provides an orientation for each of its edges
  - **they are oriented in the direction in which they are first encountered (during exploring)**
  - **the reverse direction is then ignored.**

Algorithm for depth-first search on undirected graph

```
int dfs(IntList[] adjVertices,int[] color,int v,int p,...)
  int w;
  IntList remAdj;
  int ans;

color[v]=gray;
Preorder processing of vertex v
remAdj=adjVetices[v];
while(remAdj<>nil)
  w=first(remAdj);
  if(color[w]==white)
    Explratory processing for tree edge vw.
    int wAns=dfs(adjVetices,color, w, v,....)
    BackTrack processing for tree edge vw using wAns(like inorder)
  else if(color[w]==gray && w<>p)
     Checking back edge vw
     //else wv was traversed, so ignore vw.
  remAdj=rest(remAdj)
Postorder processing of vertex v, including final computation of ans
color[v]=black;
return ans;
```

Breadth-first Search on Undirected Graph

- Simply treat the undirected graph as symmetric digraph
  - **in fact undirected graph is represented in adjacency list as symmetric digraph**
- Each edge is processed once in the “forward” direction
  - **whichever direction is encountered (explored) first is considered “forward” for the duration of the search**

Bi-connected components of an Undirected graph

- Problem:
  - **If any *one* vertex (and the edges incident upon it) are removed from a connected graph,**
  - **is the remaining subgraph still connected?**
- Biconnected graph:
  - A connected undirected graph G is said to be biconnected if it remains connected after removal of any one vertex and the edges that are incident upon that vertex.
- Biconnected component:
  - A biconnected component of a undirected graph is a maximal biconnected subgraph, that is, a binconnected subgraph not contained in any larger binconnected subgraph.
- Articulation point:
  - A vertex v is an articulation point for an undirected graph G if there are distinct vertices w and x (distinct from v also) such that v is in every path from w to x.

Bi-connected components, e.g.

- Some vertices are in more than one component (which vertices are these?)